On the Differential equations of the characters for the Renormalization group

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Abstract

Owing to the analogy between the Connes-Kreimer theory of the renormalization and the integrable systems, we derive the differential equations of the unit mass for the renormalized characters ϕ_+ and the counter term ϕ_-. We give another proof of the scattering type formula of ϕ_-. The differential equation of ϕ_- of the coordinate ε on $\mathbb{P}^1$ is also given. The hierarchy of the renormalization groups is defined as the integrable systems.

1 The Conne-Kreimer theory

We start on the same setting[1] with the Connes-Kreimer's papers [2, 3]. Let $\mathcal{H}$ be a Hopf algebra of the 1PIs of $g\phi^3$-theory. Let G be a Lie group of the characters of $\mathcal{H}$ and L the Lie algebra of derivations. The product of $\phi_1, \phi_2 \in G$ is given by

$$(\phi_1\phi_2)(X) = \langle \phi_1 \otimes \phi_2, \Delta(X)\rangle \quad \text{for any } X \in \mathcal{H}. \tag{1}$$

The inverse and a unit are defined by $\phi^{-1}(X) = \phi(S(X))$ and $1(X) = \delta_{X1}$. The Lie bracket of $\delta_1, \delta_2 \in L$ is defined as $([\delta_1, \delta_2])(X) = \langle \delta_1 \otimes \delta_2 - \delta_2 \otimes \delta_1, X\rangle$ for any $X \in \mathcal{H}$. The dual space $\mathcal{H}^*$ is an algebra with the product (1). We add the element Z_0 to L such that $\theta_t := \mathrm{Ad}\, e^{tZ_0}$ is the grading[2] of G. We define $\tilde{G} := G \rtimes_\theta \mathbb{R}$ and $\tilde{L} := L \oplus \mathbb{C}Z_0$.

We consider the loop groups[3] of $\tilde{G}$ and the loop algebra of $\tilde{L}$. Let ε be the coordinate of $\mathbb{P}^1$ for the loop group which corresponds to the parameter of the dimensional regularization. The Birkhoff decomposition in the sense of Ref. [2, 3] divides $\tilde{G}$ into $\tilde{G} = \tilde{G}_-\tilde{G}_+$, i.e., any character $\phi \in \tilde{G}$ decomposes

[1] Our notations $\phi, \phi_\pm, \mu$ in this note correspond to $\gamma, \gamma_\pm, \mu^{\frac{1}{2}}$ respectively in Ref. [2, 3].
[2] We should not confuse the degree ϕ as the loop group and the algebra $\mathcal{H}^*$.
[3] In this note, we use same notation $\tilde{G}$ (resp. $\tilde{L}$) with its loop group (resp. algebra).

uniquely as $\phi = \phi_-^{-1} \cdot \phi_+$ with the condition $\phi_- = 1$ at $\varepsilon = \infty$. The Lie algebra $\tilde{L}$ decomposes into $\tilde{L} = \tilde{L}_- \oplus \tilde{L}_+$. Keeping in mind the Feynman rules, we assume that $\phi, \phi_\pm$ depend on the coupling constant $g =: e^x$, the unite mass $\mu =: e^t$ and $\varepsilon \in \mathbb{P}^1$ such that

$$\phi = \phi(x + \varepsilon t, \varepsilon), \quad \phi_- = \phi_-(x, \varepsilon), \quad \phi_+ = \phi_+(x, t, \varepsilon). \tag{2}$$

2 The differential equations of the unit mass

The adjoint action of $e^{t\varepsilon Z_0}$ on ϕ transpose the unite mass $\mu = e^t$ of the characters $\phi_\pm$

$$\phi(x + \varepsilon t, \varepsilon) = e^{t\varepsilon Z_0} \phi(x, \varepsilon)\, e^{-t\varepsilon Z_0} = (\phi_-^{-1} \cdot \phi_+)(x, t, \varepsilon). \tag{3}$$

We define $\tilde{\phi}_+ := \phi_+\, e^{t\varepsilon Z_0} \in \tilde{G}_+$ and obtain

$$e^{t\varepsilon Z_0}\, (\phi_-^{-1} \cdot \tilde{\phi}_+)(x, 0) = (\phi_-^{-1} \cdot \tilde{\phi}_+)(x, t). \tag{4}$$

Differentiating above equation by t, we obtain equation for $\tilde{L}$

$$\frac{\partial \phi_-}{\partial t}\, \phi_-^{-1} = -(\phi_- \varepsilon Z_0 \phi_-^{-1})_-, \quad \frac{\partial \tilde{\phi}_+}{\partial t}\, \tilde{\phi}_+^{-1} = (\phi_- \varepsilon Z_0 \phi_-^{-1})_+ \tag{5}$$

where $(\cdot)_\pm$ denote the projection onto $\tilde{L}_\pm$. We assume that there exists the limit [3] $F_t(x) := \lim_{\varepsilon \to 0} (\phi_- \theta_\varepsilon \phi_-^{-1})$ and define $\beta(x) \in \tilde{L}$ such that $F_t = e^{t\beta}$. The element β satisfies [3]

$$\phi_- \left(\frac{\partial \theta_t}{\partial t} \right)_{t=0} (\phi_-^{-1}) = \phi_- \varepsilon Z_0 \phi_-^{-1} - \varepsilon Z_0 = \beta \in \tilde{L} \tag{6}$$

Then, equations (5) are

$$\frac{\partial \phi_-}{\partial t} = 0, \quad \frac{\partial \tilde{\phi}_+}{\partial t}\, \tilde{\phi}_+^{-1} = \beta + \varepsilon Z_0. \tag{7}$$

Note that these equations are equivalent to

$$\left(\varepsilon \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right) \phi_+ \cdot \phi_+^{-1} = \left(\varepsilon \frac{\partial}{\partial x} - \frac{\partial}{\partial t} \right) \phi_- \cdot \phi_-^{-1} = \beta, \tag{8}$$

since $\theta_{x'}(\phi_\pm)(x) = \phi_\pm(x + x')$. The "Baker function" [5] $w(x, t, \varepsilon) := \phi_- e^{t\varepsilon Z_0} \in \tilde{G}$ satisfies same equation with $\tilde{\phi}_+$. The first equation of (7)

means the well-known fact that ϕ_- dose not depend on unit mass. The second equation shows that ϕ_+ depend on the unit mass t as

$$\phi_+(x,t,\varepsilon) = e^{t(\beta+\varepsilon Z_0)}\phi_+(x,0,\varepsilon)\, e^{-t\varepsilon Z_0}. \tag{9}$$

For $\varepsilon = 0$, of course, (9) is reduced to $\phi_+(t) = e^{t\beta}\phi_+(0)$.

The counter term ϕ_- does not necessarily have the form $\phi_- = e^{\alpha}$ with some $\alpha \in L_-$ as the integrable systems [6]. However, it has the scattering type formula [3]

$$\phi_-(x,\varepsilon) = \lim_{t\longrightarrow\infty} e^{-t(\frac{\beta}{\varepsilon}+Z_0)}e^{tZ_0}. \tag{10}$$

Here, we give another proof than that of Ref. [3]. Owing to equation (6), we have

$$\begin{aligned} e^{-t(\frac{\beta}{\varepsilon}+Z_0)}e^{tZ_0} &= e^{-t(\phi_- Z_0 \phi_-^{-1})}e^{tZ_0} \\ &= (\phi_- e^{-tZ_0}\phi_-^{-1})e^{tZ_0} = \phi_-\theta_{-t}(\phi_-^{-1}). \end{aligned} \tag{11}$$

For $X \in \mathcal{H}$ with $\Delta(X) = \sum X' \otimes X''$, we have

$$\begin{aligned} \phi_-\theta_{-t}(\phi_-^{-1})(X) &= \langle \phi_- \otimes \phi_-^{-1}, \sum X' \otimes \theta_{-t}(X'')\rangle \\ &= \phi_-(X)\phi_-^{-1}(1) + O(e^{-t}) \longrightarrow \phi_-(X) \end{aligned} \tag{12}$$

as $t \longrightarrow \infty$. This shows the formula (10).

We can rewrite this formula in terms of the characters. Owing to (9) and (10), we have

$$e^{-tZ_0}e^{t(\frac{\beta}{\varepsilon}+Z_0)} = e^{-tZ_0}\phi_+(x,t\varepsilon^{-1},\varepsilon)e^{tZ_0}\phi_+^{-1}(x,0,\varepsilon). \tag{13}$$

Therefore we obtain

$$\phi_-^{-1}(x,\varepsilon) = \lim_{t\longrightarrow\infty}\phi_+(x-t,t\varepsilon^{-1},\varepsilon)\phi_+^{-1}(x,0,\varepsilon) \tag{14}$$

and

$$\phi(x,\varepsilon) = \lim_{t\longrightarrow\infty}\phi_+(x-t,t\varepsilon^{-1},\varepsilon). \tag{15}$$

This formulae imply that the characters ϕ, ϕ_- are recovered from ϕ_+ with the suitable limits.

3 The differential equations of the coordinate ε on $\mathbb{P}^1$

We can obtain the differential equation of ε. Differentiating equation $[Z_0, \phi_-^{-1}] = \frac{1}{\varepsilon}\beta\phi_-^{-1}$ by ε, we have

$$[Z_0, \dot{\phi}_-^{-1}\phi_-] = -\frac{1}{\varepsilon^2}\phi_-^{-1}\beta\phi_- \tag{16}$$

where $\dot{\phi}_-^{-1} = \frac{\partial}{\partial\varepsilon}(\phi_-^{-1})$. In general, if $\alpha, \alpha' \in L$ satisfies $\alpha(1) = \alpha'(1) = 0$ and $[Z_0, \alpha] = \alpha'$, α is give by the integral form [3, 4]

$$\alpha = \int_0^\infty dt\, \theta_{-t}(\alpha'). \tag{17}$$

Since $\phi_-(1) = 1$, $\beta(1) = 0$, we have $(\dot{\phi}_-^{-1}\phi_-)(1) = (\phi_-^{-1}\beta\phi_-)(1) = 0$. Therefore, by equation (16), we obtain equation

$$\frac{\partial\,\phi_-^{-1}}{\partial\,\varepsilon}\,\phi_- = -\frac{1}{\varepsilon^2}M(x,\varepsilon) \tag{18}$$

where

$$M(x,\varepsilon) = \int_0^\infty dt\, \theta_{-t}(\phi_-^{-1}\beta\phi_-) \in \tilde{L}. \tag{19}$$

If $X \in \mathcal{H}$ is a 1PI with the degree n, $M(x,\varepsilon)(X)$ is the polynomial of ε^{-1} whose highest degree is $n-1$. We can also derive the equation for the Baker function w

$$\frac{\partial w}{\partial\varepsilon}\,w^{-1} = \tilde{M}(x,t,\varepsilon), \quad \tilde{M} := \phi_-(M + tZ_0)\phi_-^{-1} \in \tilde{L}. \tag{20}$$

Note that the differential equations of ε are used in the theory of the monodromy preserving deformation of the integrable systems.

We can extend above results to the case of G_2 which is the group of the formal diffeomorphism [1, 3] of $\mathbb{C}$ $G_2 := \{\varphi \in \mathrm{Diff}(\mathbb{C}) \,|\, \varphi(z) = z + O(z^2)\ z \in \mathbb{C}\}$, with the help of the map [3] $\rho : G \longrightarrow G_2$ which is the anti-homomorphism $\rho(\phi_1\phi_2) = \rho(\phi_2) \circ \rho(\phi_1)$ of the group. For example, owing to the map ρ and equation (14), we have

$$\psi^{-1}(\varepsilon) = \lim_{t\longrightarrow\infty} \psi_+^{-1}(0,\varepsilon) \circ \alpha_{-t}(\psi_+)(t\varepsilon^{-1},\varepsilon) \tag{21}$$

where $\psi(\varepsilon) = \rho(\phi_-(0,\varepsilon))$, $\psi_+(t,\varepsilon) = \rho(\phi_+)(0,t,\varepsilon)$ and the α_t is the grading [3] of G_2 which satisfies $\alpha_t \circ \rho = \rho \circ \theta_t$.

4 The hierarchy of the renormalization group

We can introduce the hierarchy with the left action of $\exp\left(\sum_{n\geq 1} t_n \varepsilon^n Z_0\right)$ on ϕ as the integrable systems [5, 6]. These flows are simple. The ϕ_- does not depend on t_n again and $\tilde{\phi}_+ := \phi_+ e^{\sum_{n\geq 1} t_n \varepsilon^n Z_0}$ satisfies the equation $\frac{\partial \tilde{\phi}_+}{\partial t_n} \tilde{\phi}_+ = \varepsilon^{n-1}(\beta + \varepsilon Z_0)$. The $\{t_n\}$ parametrize the ε^n order flows of the renormalization.

If we enlarge the group G or G_2, the hierarchy generates the non-trivial flows. For example, we choose the Lie group whose Lie algebra $\mathcal{L}$ is generated by $\{Z_{-1}, Z_0, \{Z_T\}_T\}$ where T is the rooted tree in the sense of Ref. [1], the hierarchy is the AKNS type one associated with $\mathcal{L}$. In fact, we restrict $\mathcal{L}$ to the Lie subalgebra $\mathfrak{sl}(2, \mathbb{C})$ of $\mathcal{L}$ generated by $\{Z_{-1}, Z_0, Z_\bullet\}$ and the hierarchy corresponds to the usual AKNS hierarchy [6].

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